数理解析研究所講究録
第 1839 巻 2013 年 40-42

Semi-operator monotonicity for operator monotone functions

山形大学理学部数理科学科 佐野 隆志 (Takashi SANO)
Department of Mathematical Sciences, Faculty of Science,
Yamagata University

We review results on operator monotone functions. For details, we refer [11].

Loewner and Kwong matrices

Let $f(t)$ be a continuously differentiable function from the interval $(0,\infty)$ into itself. For distinct $t_1,\ldots,t_n$ in $(0,\infty)$, we define the $n\times n$ matrix $L_{f(t)}(t_1,\ldots,t_n)$ as

$$L_{f(t)}(t_1,\ldots,t_n):=\left[\frac{f(t_i)-f(t_j)}{t_i-t_j}\right],$$

where the diagonal entries are understood as the first derivatives $f'(t_i)$. This matrix is called a *Loewner matrix*. Similarly we define the $n\times n$ matrix $K_{f(t)}(t_1,\ldots,t_n)$ as

$$K_{f(t)}(t_1,\ldots,t_n):=\left[\frac{f(t_i)+f(t_j)}{t_i+t_j}\right],$$

which we call an *Kwong matrix*. (In [2, 8] it is called an anti-Loewner matrix.) See [3, 4, 5] on Loewner and Kwong matrices.

We also define the $n\times n$ matrix $L_{f(t)}^{(m)}(t_1,\ldots,t_n)$ and $K_{f(t)}^{(m)}(t_1,\ldots,t_n)$ as

$$\begin{aligned}
L_{f(t)}^{(m)}(t_1,\ldots,t_n) &:= \left[\frac{\{f(t_i)\}^m-\{f(t_j)\}^m}{t_i^m-t_j^m}\right],\\
K_{f(t)}^{(m)}(t_1,\ldots,t_n) &:= \left[\frac{\{f(t_i)\}^m+\{f(t_j)\}^m}{t_i^m+t_j^m}\right]
\end{aligned}$$

for a positive integer m.

It is well-known that $f(t)$ is operator monotone if and only if for all n and $t_1,\ldots,t_n$, the Loewner matrices $L_{f(t)}(t_1,\ldots,t_n)$ are positive semidefinite; see [10]. If $f(t)$ is operator monotone, the Kwong matrices $K_{f(t)}(t_1,\ldots,t_n)$ are positive semidefinite; see [9]. The latter is recently characterized by

Audenaert [2]. On the other hand, it is known that if $f(t)$ is operator monotone, so is the function $t \mapsto \{f(t^{1/m})\}^m$ for any positive integer m. See [1, 7]. Hence, combining them, we see that if f is operator monotone, then the Loewner matrices $L_{\{f(t^{1/m})\}^m}(t_1,\ldots,t_n)$ and the Kwong matrices $K_{\{f(t^{1/m})\}^m}(t_1,\ldots,t_n)$ are positive semidefinite; therefore, so are $L^{(m)}_{f(t)}(t_1,\ldots,t_n)$ and $K^{(m)}_{f(t)}(t_1,\ldots,t_n)$.

We have an alternative proof for operator monotonicity of the function $t \mapsto \{f(t^{1/m})\}^m$ by Theorem 1.6 that if f is operator monotone, then $L^{(m)}_{f(t)}(t_1,\ldots,t_n)$ are positive semidefinite for all n and $t_1,\ldots,t_n$ in $(0,\infty)$. We also have in Theorem 1.5 that if f is operator monotone, then $K^{(m)}_{f(t)}(t_1,\ldots,t_n)$ are positive semidefinite for all n and $t_1,\ldots,t_n$ in $(0,\infty)$.

We recall several facts:

Theorem 1.1 (Löwner [10]) Let f be a C^1 function on $(0,\infty)$. Then f is operator monotone if and only if $L_{f(t)}(t_1,\ldots,t_n)$ are positive semidefinite for all positive integers n and $t_1,\ldots,t_n$ in $(0,\infty)$.

Theorem 1.2 (Kwong [9]) Let f be a positive C^1 function on $(0,\infty)$. If f is operator monotone, then $K_{f(t)}(t_1,\ldots,t_n)$ are positive semidefinite for all positive integers n and $t_1,\ldots,t_n$ in $(0,\infty)$.

Theorem 1.3 (Audenaert [2]) Let f be a positive C^1 function on $(0,\infty)$. For all positive integers n and $t_1,\ldots,t_n$ in $(0,\infty)$ $K_{f(t)}(t_1,\ldots,t_n)$ are positive semidefinite if and only if $f(\sqrt{t})\sqrt{t}$ is operator monotone.

Theorem 1.4 (Ando [1], Fujii-Fujii [7]) Let f be an operator monotone function from $(0,\infty)$ into itself. Then so is the function $t \mapsto \{f(t^{1/m})\}^m$ for any positive integer m.

We have the following theorems in [11]:

Theorem 1.5 Let f be an operator monotone function from $(0,\infty)$ into itself. Then for any positive integer m, $K^{(m)}_{f(t)}(t_1,\ldots,t_n)$ are positive semidefinite for all positive integers n and $t_1,\ldots,t_n$ in $(0,\infty)$: or $K_{\{f(t^{1/m})\}^m}(t_1,\ldots,t_n)$ are positive semidefinite for all positive integers n and $t_1,\ldots,t_n$ in $(0,\infty)$.

Theorem 1.6 Let f be an operator monotone function from $(0,\infty)$ into itself. Then for any positive integer m, $L^{(m)}_{f(t)}(t_1,\ldots,t_n)$ are positive semidefinite for all positive integers n and $t_1,\ldots,t_n$ in $(0,\infty)$: or $L_{\{f(t^{1/m})\}^m}(t_1,\ldots,t_n)$ are positive semidefinite for all positive integers n and $t_1,\ldots,t_n$ in $(0,\infty)$.

参考文献

[1] T. Ando, *Concavity of certain maps on positive definite matrices and applications to Hadamard products,* Linear Algebra Appl., 26 (1979), 203-241.

[2] K. M. R. Audenaert, *A characterisation of anti-Löwner functions,* Proc. Amer. Math. Soc., 139 (2011), 4217-4223.

[3] R. Bhatia, *Positive Definite Matrices,* Princeton University Press (2007).

[4] R. Bhatia and T. Sano, *Loewner matrices and operator convexity,* Math. Ann., 344 (2009), 703-716.

[5] R. Bhatia and T. Sano, *Positivity and conditional positivity of Loewner matrices,* Positivity 14 (2010), 421-430.

[6] W. F. Donoghue, *Monotone Matrix Functions and Analytic Continuation,* Springer (1974).

[7] J. I. Fujii and M. Fujii, *An analogue to Hansen's theory of generalized Löwner's functions,* Math. Japonica, 35 (1990), 327-330.

[8] C. Hidaka and T. Sano, *Conditional negativity of anti-Loewner matrices,* Linear and Multilinear Algebra, 60 (2012), 1265-1270.

[9] M. K. Kwong, *Some results on matrix monotone functions,* Linear Algebra Appl., 118 (1989), 129-153.

[10] K. Löwner, *Über monotone Matrixfunctionen,* Math. Z., 38 (1934), 177-216.

[11] S. Tachibana and T. Sano, *On Loewner and Kwong matrices,* to appear in Sci. Math. Japonicae.